Close Thu: 10.1/13.1
Close next Tue: 10.2/13.2, 10.3
Close next Thu: 13.3 (finish much sooner)
Midterm 1, Thursday, Apr. 20th
Covers 12.1-12.5, 10.1-10.3, 13.1-13.3

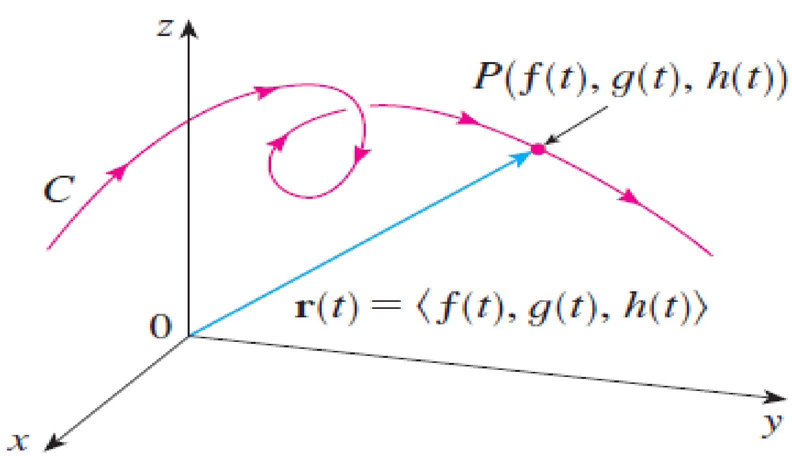
10.1/13.1,10.2/13.2: 3D Curves

Parametric equations:

$$x = x(t), y = y(t), z = z(t)$$

Vector form:

$$\vec{\boldsymbol{r}}(t) = \langle x(t), y(t), z(t) \rangle$$



Eliminating the parameter

To get the surface/path over which the motion is occurring:

(a) Solving for t, substitute.

(b) Use $(\sin(u))^2 + (\cos(u))^2 = 1$.

Entry Task:

Eliminate the parameters:

1. $x = t, y = 2 - t^2$

2. $x = 3\cos(4t), y = 4\sin(4t)$

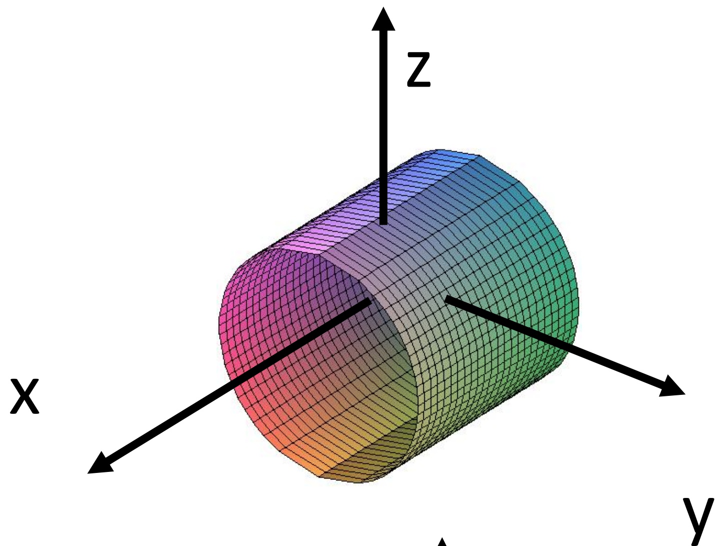
3. $x = t, y = \cos(2t), z = \sin(2t)$

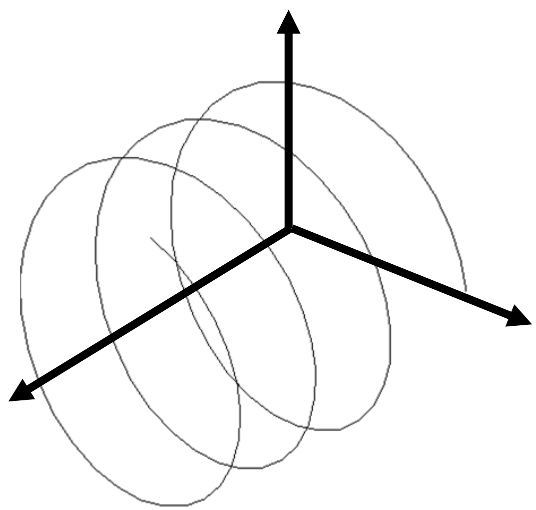
4. $x = t\cos(t), y = t\sin(t), z = t$

All pts given by the equations:

$$x = t\,, y = \cos(2t)\ \ , z = \sin(2t)$$

are on the cylinder: $y^2\ +\ z^2\ =\ 1.$

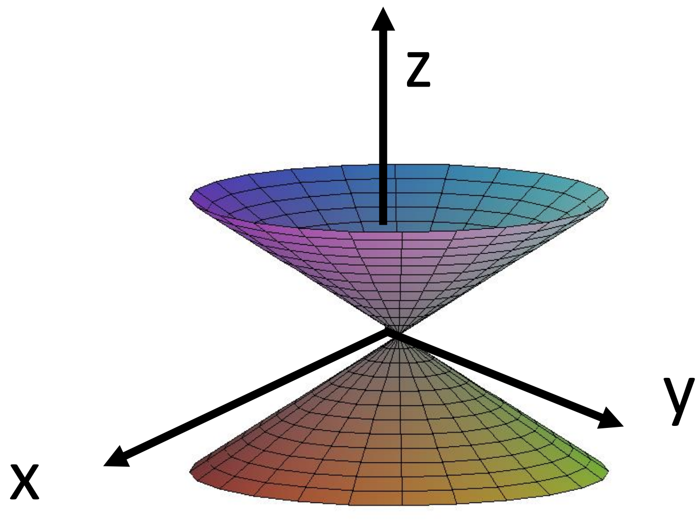


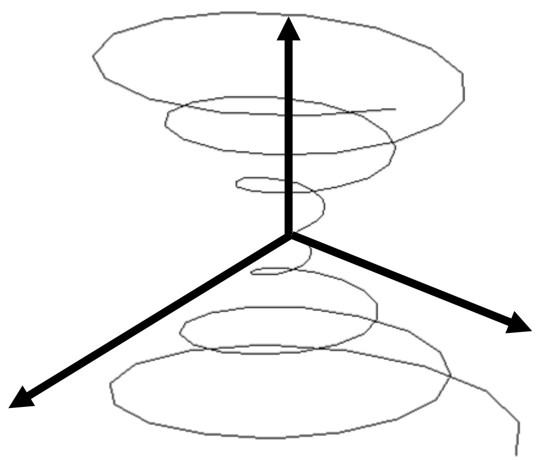


All pts given by the equations:

$$x = t\cos(t)\,, y = t\sin(t)\,, z = t$$

are on the cone: $z^2 = x^2 + y^2$.





Intersection issues

For all intersection questions, combine/substitute the conditions!

(a) ***Intersecting a curve and surface***.
Combine conditions!

Example:
Find all intersections of

$$x = t, y = \cos(\pi\, t)\,, z = \sin(\pi\, t)$$

with the surface

$$x^2 - y^2 - z^2 = -3.$$

(b) ***Intersecting two curve***.

Use two different parameters!!!

Combine conditions!

We say the objects **collide** if the intersection happens at the same parameter value (i.e. same time).

Example:

Two particles are moving according to

$$\boldsymbol{r}_1(t) = \langle t, 5t, 9 \rangle, \text{and}$$
$$\boldsymbol{r}_2(t) = \langle t - 2, 5, t^2 \rangle.$$

Do their paths intersect?

Do they collide?

(c) ***Intersecting two surfaces***.

Combine conditions!

Answer will be a 3D curve.

i) Let one variable be t

ii) Solve for others in terms of t.

(or if $x^2 + y^2 = 1$, use

$x = cos(t), y = sin(t)$)

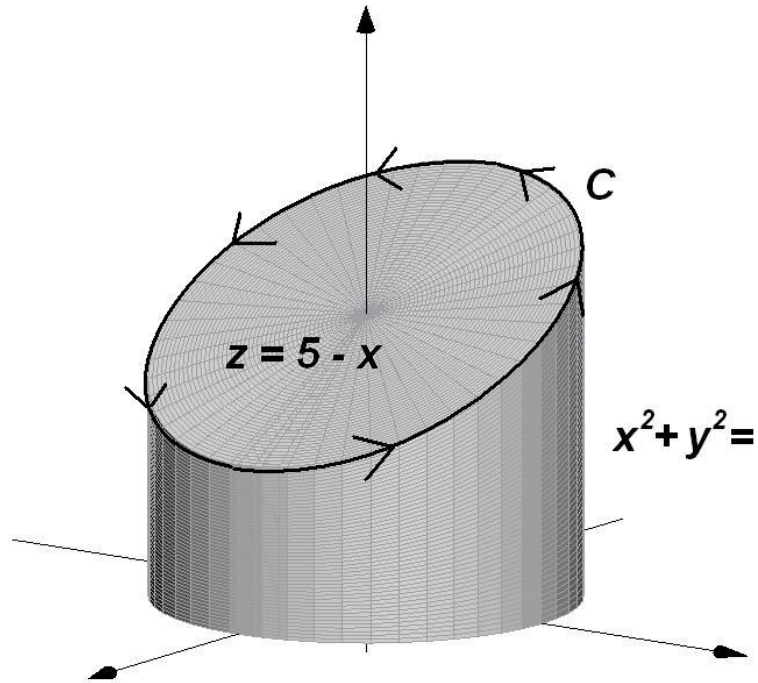
Example:

Find *any* parametric equations that describe the curve of intersection of

$$z^2 = x^2 + y^2 \text{ and } z = 2y$$

Find *any* parametric equations that describe the curve of intersection of

$$1 = x^2 + y^2 \text{ and } z = 5 - x.$$



Calculus on curves

Review: Going from parametric to slope and concavity in 2D:

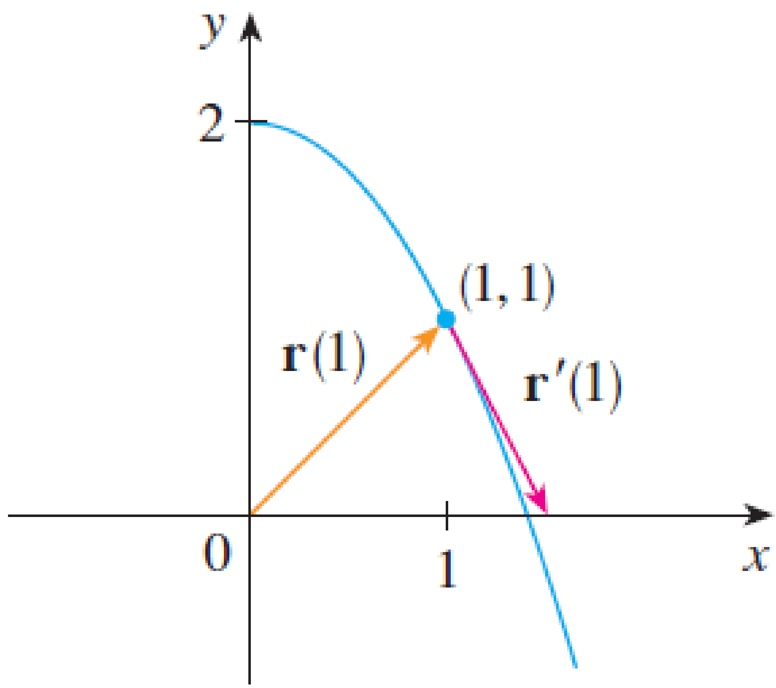
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\big(f'(x)\big)}{dx/dt}$$

Example: Consider

$$x = t, y = 2 - t^2$$

Example: Consider

$$\boldsymbol{r}(t) = \langle t, 2 - t^2 \rangle$$



y
2
(1, 1)
r(1)
r′(1)
0
1
x

Vector Calculus

If $\vec{\boldsymbol{r}}(t) = \langle x(t), y(t), z(t)\rangle$, define

$\vec{\boldsymbol{r}}'(t) =$

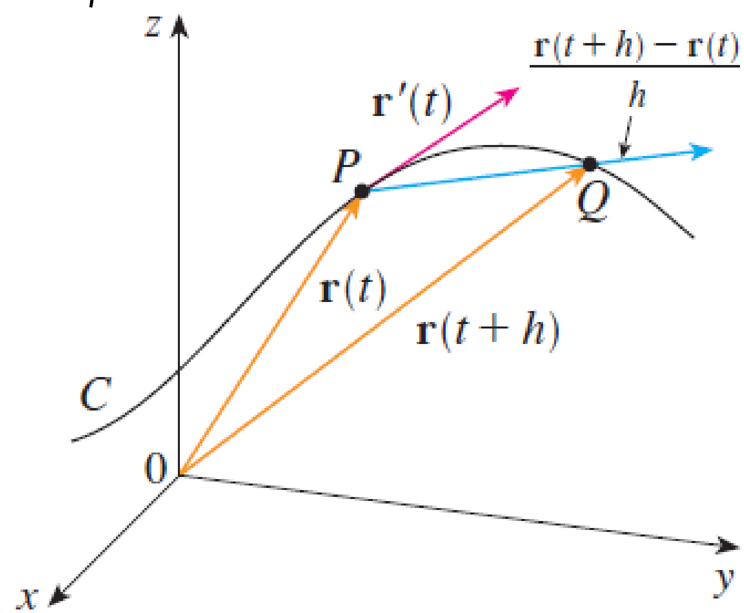
$$\lim_{h\to 0}\left\langle \frac{x(t+h)-x(t)}{h}, \frac{y(t+h)-y(t)}{h}, \frac{z(t+h)-z(t)}{h}\right\rangle$$

so $\quad \vec{\boldsymbol{r}}'(t) = \langle x'(t), y'(t), z'(t)\rangle.$

We also define

$$\vec{\boldsymbol{r}}''(t) = \langle x''(t), y''(t), z''(t)\rangle.$$

$$\int \vec{\boldsymbol{r}}(t)dt = \left\langle \int x(t)dt, \int y(t)dt, \int z(t)dt \right\rangle$$



Morale, do calculus **component-wise**.

In 13.3, $\vec{\boldsymbol{r}}''(t)$ gives curvature info.
In 13.4, $\vec{\boldsymbol{r}}'(t)$ is velocity, $|\vec{\boldsymbol{r}}'(t)|$ is speed, and $\vec{\boldsymbol{r}}''(t)$ is acceleration.

Consider $\vec{\boldsymbol{r}}(t) = \langle t, \cos(2t), \sin(2t) \rangle$.

(a) Find $\vec{\boldsymbol{r}}'(t)$, $|\vec{\boldsymbol{r}}'(t)|$, and $\vec{\boldsymbol{r}}''(t)$.

(b) Find $\vec{\boldsymbol{r}}(\pi/4)$ and $\vec{\boldsymbol{r}}'(\pi/4)$.

(c) Give the equation for the tangent line at $t = \pi/4$

Arc Length

The length of a curve
from $t = a$ to $t = b$ is given by

$$\int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}\,dt$$

$$= \int_a^b |\vec{\boldsymbol{r}}'(t)|dt$$

(Note: 2D is same without the $z'(t)$).
We call this **arc length**.

The arc length from 0 to u is often written as

$$s(u) = \int_0^u |\vec{\boldsymbol{r}}'(t)|dt$$

We call this the **arc length function**.